

# Evolution equations for the perturbations of slowly rotating relativistic stars

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## ABSTRACT

We present a new derivation of the equations governing the oscillations of slowly rotating relativistic stars. Previous investigations have been mostly carried out in the Regge–Wheeler gauge. However, in this gauge the process of linearizing the Einstein field equations leads to perturbation equations which as such cannot be used to perform numerical time evolutions. It is only through the tedious process of combining and rearranging the perturbation variables in a clever way that the system can be cast into a set of hyperbolic first order equations, which is then well suited for the numerical integration. The equations remain quite lengthy, and we therefore rederive the perturbation equations in a different gauge, which has been first proposed by Battiston et al. (1970). Using the ADM formalism, one is immediately lead to a first order hyperbolic evolution system, which is remarkably simple and can be numerically integrated without many further manipulations. Moreover, the symmetry between the polar and the axial equations becomes directly apparent.

**Key words:** relativity – methods: numerical – stars: neutron – stars: oscillations – stars: rotation

## 1 INTRODUCTION

The theory of non-radial perturbations of relativistic stars has been a field of intensive study for more than three decades, beginning with the pioneering paper of Thorne & Campolattaro in 1967. Their work was in turn based on previous studies of black hole perturbations initiated by Regge & Wheeler in 1957. Because of the quite involved and tedious computations of the perturbed field equations, the main focus remained on non-rotating neutron stars, although the foundations for computing rotating relativistic stellar models have already been laid by Hartle in 1967. Only in the early 90s, work began to shift to the perturbations of rotating relativistic neutron stars. In most works, the slow-rotation approximation was used to tackle the problem. Chandrasekhar & Ferrari (1991) studied the axisymmetric perturbations, where they established the coupling between the polar and axial modes induced by the rotation (polar or even parity modes are characterised by a sign change under parity transformation according to  $(-1)^l$  while the axial ones change as  $(-1)^{l+1}$ ). Shortly later, Kojima (1992) presented the first complete derivation of the coupled polar and axial perturbation equations. These equations were the starting point for investigations of relativistic rotational effects on stellar oscillations and associated instabilities. Most of the work considered the simpler task of solving the perturbation equations in the frequency domain and, as a result, calculations involved the determination of the eigenfrequencies rather than the solution of the time dependent equations. A somewhat different approach based on a Lagrangian description was used by Lockitch et al. (2001) with the focus on the computation of rotationally induced inertial modes.

Following the more than 40 years old tradition, it was quite common to work in the Regge–Wheeler gauge, although some groups were using different gauges or the gauge invariant formulation of Moncrief (1974). In particular, a gauge introduced by Battiston, Cazzola & Lucaroni in 1971 in a series of papers to study the stability properties of non-radial oscillations in relativistic non-rotating stars, has not received proper attention (Battiston, Cazzola & Lucaroni 1971, Cazzola & Lucaroni 1972, 1974, 1978; Cazzola, Lucaroni & Semezato 1978).

Since the perturbation equations of non-rotating stars are fairly simple, there is no real advantage of one gauge over the other. Note, however, that there was a long standing puzzle why in the Regge–Wheeler gauge the equations could be reduced

to a fourth order system, whereas in the diagonal gauge used by Chandrasekhar & Ferrari (1991), which was previously used by Chandrasekhar (1983) in the context of black hole perturbations, only a fifth order system could be obtained. The latter system gave rise to an additional divergent solution, which had to be rejected on physical grounds. This discrepancy was finally solved by Price & Ipser (1991), who showed that the diagonal gauge was not completely fixed and still possessed one degree of gauge freedom, giving rise to the additional spurious solution.

As computer power has been enormously increasing within the last decade, the problem of evolving the fully non-linear 3D Einstein equations in the time domain has finally become into reach of feasibility. Various groups around the world are now building robust codes that perform the time evolution of both single neutron stars or binaries in full 3D (see for instance Font et al. 2001, Shibata & Uryu 2001, and the review of Stergioulas 1998). Nevertheless, there is still considerable work in progress within perturbative approaches, which can provide us with a deeper understanding of the physics of relativistic star perturbations. Moreover, any trustable non-linear code must be able to reproduce the results from perturbation theory.

Time evolutions of the perturbative equations have been carried out first for the axial equations (Andersson & Kokkotas 1996) and then for the polar equations using the Regge–Wheeler gauge by Allen et al. (1998) and Ruoff (2001a). Allen et al. (1998) managed to write down the evolution equations as two relatively simple wave equations for the metric perturbations and one wave equation for the fluid enthalpy perturbation inside the star. Ruoff (2001a) rederived these equations using the ADM formalism (Arnowitt et al. 1962). They were used to evolve and study initial data, representing the late stage of a binary neutron star head-on collision (Allen et al. 1999).

Using the Arnowitt–Deser–Misner (ADM) formalism, the evolution equations for the axial perturbations of rotating stars have been brought into a suitable form for the numerical evolution by Ruoff & Kokkotas (2001a,b). Here, the resulting system of equations came out immediately as a first order system both in space and time, and it could be directly used for the numerical evolution without many further manipulations. In the non-rotating case, it is an easy task to transform the first order system into a single wave equation for just one metric variable. In the rotating case, however, this is not possible any more because of the rotational correction terms.

When looking at the set of polar equations derived by Kojima (1992) it appears clearly that the presence of mixed spatial and time derivatives makes them not suitable for the numerical time integration. Nevertheless, using a number of successive manipulations and the introduction of many additional variables, we were able to cast the equations into a hyperbolic first order form.

A more natural way to automatically obtain a set of equations, which is first order in time, is to use the ADM formalism. However, as we shall explain, even in that case the polar equations in the Regge–Wheeler gauge need to be further manipulated before they are suitable for a numerical integration. In general the ADM formalism yields a set of partial differential equations which are first order in time, but second order in space. For the numerical evolution, this is not ideal and one would rather prefer to have a pure first order system, or if possible a pure second order system, thus representing generalized wave equations. As we mentioned above, in the non-rotating case, it is easily possible to convert the perturbation equations into a set of wave equations. However, in the rotating case, this is not possible any more, even in the simple case when only axial perturbations are considered. To illustrate the problems associated with the Regge–Wheeler gauge, let us recall Einstein’s (unperturbed) evolution equations written in the ADM formalism:

$$(\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2\alpha K_{ij}, \quad (1)$$

$$(\partial_t - \mathcal{L}_\beta) K_{ij} = -\alpha_{;ij} + \alpha [R_{ij} + K^k_k K_{ij} - 2K_{ik} K^k_j - 4\pi (2T_{ij} - T^\nu_\nu \gamma_{ij})], \quad (2)$$

with  $\alpha$  denoting the lapse function,  $\beta^i$  the shift vector,  $\mathcal{L}_\beta$  the Lie-derivative with respect to  $\beta^i$ ,  $\gamma_{ij}$  the metric of a space-like three dimensional hypersurface with Ricci tensor  $R_{ij}$ , and  $K_{ij}$  its extrinsic curvature. It is obvious that the only second order spatial derivatives are  $\partial_i \partial_j \alpha$  and  $\partial_i \partial_j \gamma_{kl}$  with the latter coming from the Ricci tensor  $R_{ij}$ . This is still true for the linearized version of Eqs. (1) and (2).

In the Regge–Wheeler gauge, we have a non-vanishing perturbation of the lapse  $\alpha$  and of the diagonal components of the spatial perturbations  $h_{ij}$ . Using the notation of Ruoff (2001a), the polar perturbations can be written as

$$\alpha \sim \sum_{l,m} S_1^{lm}(t,r) Y_{lm}(\theta, \phi), \quad (3)$$

$$h_{ij} \sim \sum_{l,m} \begin{pmatrix} S_3^{lm}(t,r) & 0 & 0 \\ 0 & T_2^{lm}(t,r) & 0 \\ 0 & 0 & \sin^2 \theta T_2^{lm}(t,r) \end{pmatrix} Y_{lm}(\theta, \phi). \quad (4)$$

The perturbation equations coming from Eq. (2) contain second order  $r$ -derivatives of  $S_1$  and  $T_2$ . Note that they do not contain second order derivatives of  $S_3$ , because only the angular components of the metric get differentiated twice with respect to  $r$ . In the axial case there is only one perturbation function for the angular metric components, but it is set to zero in the Regge–Wheeler gauge. Therefore the ADM formalism immediately yields a first order system.

The polar equations, in contrast, can be cast into a fully first order system only through the introduction of some auxiliary

variables. In the non-rotating case, this is a fairly easy task, but for the rotating case, it turns out to become considerably more complicated. This is because the simple proportionality of  $S_1$  and  $S_3$ , which holds in the non-rotating case and which makes it easy to replace  $S_1$  by  $S_3$ , does not do so any more in the rotating case. Instead this relation involves rotational correction terms, and the replacement of  $S_1$  by  $S_3$  would lead to a considerable inflation of the equations.

Consequently, instead of manipulating the perturbation equations in the Regge–Wheeler gauge, we look for a gauge in which the perturbation equations, by construction, do not show any second order spatial derivative. We have seen that the second derivatives come from the angular terms in the spatial metric and the lapse function. It seems therefore natural to set these components to zero. For the axial case this is already realized in the Regge–Wheeler gauge. It is only for the polar perturbations that we need a different gauge. From the seven polar components of the metric, Regge & Wheeler (1957) chose to set the components  $V_1$ ,  $V_3$  and  $T_1$ , which represent in the notation of Ruoff (2001a) the polar angular vector perturbations and one of the angular tensor perturbations, to zero. We now choose a different set, namely we set the angular terms in the spatial metric  $T_1$ ,  $T_2$  together with the lapse  $S_1$  to zero and retain  $V_1$  and  $V_3$ . With this choice we expect the ADM formalism to provide us with an evolution system without any second  $r$ -derivatives.

We should mention again that this gauge has actually been introduced thirty years ago by Battiston, Cazzola & Lucaroni (1971) to derive the perturbation equations for non-radial oscillations of non-rotating neutron stars and to investigate in a subsequent series of papers their stability properties (Cazzola & Lucaroni 1972, 1974, 1978; Cazzola, Lucaroni & Semezato 1978a, 1978b). Particularly relevant is the first paper of the series, in which they proved the uniqueness of this gauge, hereafter referred to as the BCL gauge, and where they also showed the relation with the Regge–Wheeler gauge.

In Section 2, we will use the ADM formalism to derive the time dependent perturbation equations for slowly rotating relativistic stars in the BCL gauge. Section 3 contains a brief discussion of the non-rotating limit and conclusions will be given in Section 4. In the Appendix, we present the perturbation equations as they follow directly from Einstein’s equations in a form similar to the equations in the Regge–Wheeler gauge given by Kojima (1992). Throughout the paper, we adopt the metric signature  $(-+++)$ , and we use geometrical units with  $c = G = 1$ . Derivatives with respect to the radial coordinate  $r$  are denoted by a prime, while derivatives with respect to time  $t$  are denoted by an over-dot. Greek indices run from 0 to 3, Latin indices from 1 to 3.

## 2 THE PERTURBATION EQUATIONS IN THE ADM FORMALISM

The metric describing a slowly rotating neutron star reads in spherical coordinates  $(t, r, \theta, \phi)$

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu} & 0 & 0 & -\omega r^2 \sin^2 \theta \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\omega r^2 \sin^2 \theta & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad (5)$$

where  $\nu$ ,  $\lambda$  and the “frame dragging”  $\omega$  are functions of the radial coordinate  $r$  only. With the neutron star matter described by a perfect fluid with pressure  $p$ , energy density  $\epsilon$ , and 4-velocity

$$U^\mu = (e^{-\nu}, 0, 0, \Omega e^{-\nu}) , \quad (6)$$

the Einstein equations together with an equation of state  $p = p(\epsilon)$  yield the well known TOV equations plus an extra equation for the frame dragging. To linear order, it is given by

$$\varpi'' - \left( 4\pi r e^{2\lambda} (p + \epsilon) - \frac{4}{r} \right) \varpi' - 16\pi e^{2\lambda} (p + \epsilon) \varpi = 0 , \quad (7)$$

where

$$\varpi := \Omega - \omega \quad (8)$$

represents the angular velocity of the fluid relative to the local inertial frame. In the language of the ADM formalism, we have to express the above background metric (5) in terms of lapse, (covariant) shift and the 3-metric, which we denote by  $A$ ,  $B_i$  and  $\gamma_{ij}$ , respectively. Explicitly, we have

$$A = \sqrt{B^i B_i - g_{00}} = e^\nu + O(\omega^2) , \quad (9)$$

$$B_i = (0, 0, -\omega r^2 \sin^2 \theta) , \quad (10)$$

$$\gamma_{ij} = \begin{pmatrix} e^{2\lambda} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} . \quad (11)$$

The extrinsic curvature of the space-like hypersurface described by  $\gamma_{ij}$  can be computed by

$$K_{ij} = \frac{1}{2A} (B^k \partial_k \gamma_{ij} + \gamma_{ki} \partial_j B^k + \gamma_{kj} \partial_i B^k) , \quad (12)$$

yielding the only non-vanishing components

$$K_{13} = K_{31} = -\frac{1}{2} \omega' e^{-\nu} r^2 \sin^2 \theta . \quad (13)$$

The perturbations of the background lapse  $A$ , shift  $B_i$ , 3-metric  $\gamma_{ij}$ , extrinsic curvature  $K_{ij}$ , 4-velocity  $U_i$ , energy density  $\epsilon$  and pressure  $p$  will be denoted by  $\alpha$ ,  $\beta_i$ ,  $h_{ij}$ ,  $k_{ij}$ ,  $u_i$ ,  $\delta\epsilon$  and  $\delta p$ , respectively. The twelve evolution equations for  $h_{ij}$  and  $k_{ij}$  are obtained by linearizing the non-linear ADM equations (1) and (2). Working in the slow-rotation approximation, we keep only terms up to order  $\Omega$  (or  $\omega$ ). The background quantities  $B^k$  and  $K_{ij}$  are first order in  $\Omega$ , hence we can neglect any products thereof. Furthermore, it is  $K = \gamma^{ij} K_{ij} = 0$ . These circumstances lead to cancellations of various terms and the perturbations equations reduce to:

$$\partial_t h_{ij} = \partial_i \beta_j + \partial_j \beta_i - 2 (A k_{ij} + K_{ij} \alpha + \Gamma_{ij}^k \beta_k + B_k \delta \Gamma_{ij}^k) , \quad (14)$$

$$\begin{aligned} \partial_t k_{ij} = & \alpha [R_{ij} + 4\pi(p - \epsilon)\gamma_{ij}] - \partial_i \partial_j \alpha + \Gamma_{ij}^k \partial_k \alpha + \delta \Gamma_{ij}^k \partial_k A \\ & + A [\delta R_{ij} + K_{ij} k - 2 (K_i^k k_{jk} + K_j^k k_{ik}) + 4\pi((p - \epsilon)h_{ij} + \gamma_{ij}(\delta p - \delta\epsilon) - 2(p + \epsilon)(u_i \delta u_j + u_j \delta u_i))] \\ & + B^k \partial_k k_{ij} + (\partial_k K_{ij} - K_i^l \partial_j \gamma_{kl} - K_j^l \partial_i \gamma_{kl}) \beta^k + k_{ik} \partial_j B^k + k_{jk} \partial_i B^k + K_i^k \partial_j \beta_k + K_j^k \partial_i \beta_k , \end{aligned} \quad (15)$$

where

$$k := \gamma^{ij} k_{ij} , \quad (16)$$

$$\delta \Gamma_{ij}^k := \frac{1}{2} \gamma^{km} (\partial_i h_{mj} + \partial_j h_{mi} - \partial_m h_{ij} - 2\Gamma_{ij}^l h_{lm}) , \quad (17)$$

$$\delta R_{ij} := \partial_k \delta \Gamma_{ij}^k - \partial_j \delta \Gamma_{ik}^k + \Gamma_{ij}^l \delta \Gamma_{lk}^k + \Gamma_{lk}^k \delta \Gamma_{ij}^l - \Gamma_{ik}^l \delta \Gamma_{lj}^k - \Gamma_{lj}^k \delta \Gamma_{ik}^l . \quad (18)$$

To obtain a closed set of evolution equations, we will also use the four evolution equations the fluid perturbations following from the linearized conservation law  $\delta T^{\mu\nu}_{;\mu} = 0$ . Last but not least we need the four linearized constraint equations, which serve to construct physically valid initial data and to monitor the accuracy of the numerical evolution:

$$\gamma^{ij} \delta R_{ij} - h^{ij} R_{ij} - 2K^{ij} k_{ij} = 16\pi (\delta\epsilon + 2e^{-\nu}(p + \epsilon)(\Omega - \omega)\delta u_3) , \quad (19)$$

$$\begin{aligned} -8\pi [(p + \epsilon)\delta u_i + u_i (\delta p + \delta\epsilon)] = & \gamma^{jk} (\partial_i k_{jk} - \partial_j k_{ik} - \Gamma_{ik}^l k_{jl} + \Gamma_{jk}^l k_{il} - \delta \Gamma_{ik}^l K_{jl} + \delta \Gamma_{jk}^l K_{il}) \\ & - h^{jk} (\partial_i K_{jk} - \partial_j K_{ik} - \Gamma_{ik}^l K_{jl} + \Gamma_{jk}^l K_{il}) . \end{aligned} \quad (20)$$

We assume the oscillations to be adiabatic, thus the relation between the Eulerian pressure perturbation  $\delta p$  and density perturbation  $\delta\epsilon$  is given by

$$\delta p = \frac{\Gamma_1 p}{p + \epsilon} \delta\epsilon + p' \xi^r \left( \frac{\Gamma_1}{\Gamma} - 1 \right) , \quad (21)$$

where  $\Gamma_1$  represents the adiabatic index of the perturbed configuration,  $\Gamma$  is the background adiabatic index

$$\Gamma = \frac{p + \epsilon}{p} \frac{dp}{d\epsilon} , \quad (22)$$

and  $\xi^r$  is the radial component of the fluid displacement vector  $\xi^\mu$ . The latter is related to the (covariant) 4-velocity perturbations  $\delta u_\mu$  as follows

$$\delta u_\mu = u^\nu h_{\mu\nu} + g_{\mu\nu} u^\lambda \frac{\partial \xi^\nu}{\partial x^\lambda} - \frac{1}{2} u_\mu u^\kappa u^\lambda h_{\kappa\lambda} . \quad (23)$$

For the  $r$  component, this gives us

$$(\partial_t + \Omega \partial_\phi) \xi^r = e^{-2\lambda} (e^\nu \delta u_r - \beta_r - \Omega h_{r\phi}) . \quad (24)$$

To proceed further, we expand the complete set of perturbations variables into spherical harmonics  $Y_{lm} = Y_{lm}(\theta, \phi)$ . This will enable us to eliminate the angular dependence and obtain a set of equations for the coefficients, which now only depend on  $t$  and  $r$ . It is only then that we can finally choose our gauge. In principle, choosing the gauge amounts to providing prescriptions for lapse  $\alpha$  and shift  $\beta_i$ . Those so-called slicing conditions determine how the space-like 3-metric foliates the 4-dimensional spacetime. In perturbation theory, the gauge can be used to set some of the ten metric perturbations to zero. We could, for instance, set  $\alpha = \beta_i = 0$ , and we would be left with only the six components  $h_{ij}$ . Note, that setting  $\alpha$  to zero is possible, since this is only the perturbation of the background lapse  $A$ , and the latter does not vanish.

However, our actual goal is to set some of the spatial perturbation components  $h_{ij}$  to zero, in particular the angular components  $h_{ab}$  with  $a, b = \{\theta, \phi\}$ . In principle we can prescribe the values of  $h_{ij}$  only once for the initial data, and not

throughout the evolution. The only possible way to keep  $h_{ab}$  zero throughout the evolution is to choose our gauge such that the evolution equations for  $h_{ab}$  become trivial, i.e. we have to have

$$\partial_t h_{ab} = 0, \quad a, b = \{\theta, \phi\}. \quad (25)$$

We will see that this requirement leads to a unique algebraic condition for the shift vector  $\beta_i$ . With the definitions

$$X_{lm} := 2(\partial_\theta - \cot \theta) \partial_\phi Y_{lm}, \quad (26)$$

$$W_{lm} := \left( \partial_{\theta\theta}^2 - \cot \theta \partial_\theta - \frac{\partial_{\phi\phi}^2}{\sin^2 \theta} \right) Y_{lm} = (2\partial_{\theta\theta}^2 + l(l+1)) Y_{lm}, \quad (27)$$

we now expand the metric as follows. For the polar part we choose (symmetric components are denoted by an asterisk)

$$\alpha = 0, \quad (28)$$

$$\beta_i^{polar} = \sum_{l,m} (e^{2\lambda} S_2^{lm}, V_1^{lm} \partial_\theta, V_1^{lm} \partial_\phi) Y_{lm}, \quad (29)$$

$$h_{ij}^{polar} = \sum_{l,m} \begin{pmatrix} e^{2\lambda} S_3^{lm} & V_3^{lm} \partial_\theta & V_3^{lm} \partial_\phi \\ \star & 0 & 0 \\ \star & 0 & 0 \end{pmatrix} Y_{lm}, \quad (30)$$

and the axial part is

$$\beta_i^{axial} = \sum_{l,m} \left( 0, -V_2^{lm} \frac{\partial_\phi}{\sin \theta}, V_2^{lm} \sin \theta \partial_\theta \right) Y_{lm}, \quad (31)$$

$$h_{ij}^{axial} = \sum_{l,m} \begin{pmatrix} 0 & -V_4^{lm} \frac{\partial_\phi}{\sin \theta} & V_4^{lm} \sin \theta \partial_\theta \\ \star & 0 & 0 \\ \star & 0 & 0 \end{pmatrix} Y_{lm}. \quad (32)$$

For the extrinsic curvature we have all six components

$$k_{ij}^{polar} = \frac{1}{2} e^{-\nu} \times \sum_{l,m} \begin{pmatrix} e^{2\lambda} K_1^{lm} Y_{lm} & e^{2\lambda} K_2^{lm} \partial_\theta Y_{lm} & e^{2\lambda} K_2^{lm} \partial_\phi Y_{lm} \\ \star & (r K_4^{lm} - \Lambda K_5^{lm}) Y_{lm} + K_5^{lm} W_{lm} & K_5^{lm} X_{lm} \\ \star & K_5^{lm} X_{lm} & \sin^2 \theta [(r K_4^{lm} - \Lambda K_5^{lm}) Y_{lm} - K_5^{lm} W_{lm}] \end{pmatrix}, \quad (33)$$

$$k_{ij}^{axial} = \frac{1}{2} e^{-\nu} \sum_{l,m} \begin{pmatrix} 0 & -e^{2\lambda} K_3^{lm} \frac{\partial_\phi Y_{lm}}{\sin \theta} & e^{2\lambda} K_3^{lm} \sin \theta \partial_\theta Y_{lm} \\ \star & -K_6^{lm} \frac{X_{lm}}{\sin \theta} & K_6^{lm} \sin \theta W_{lm} \\ \star & K_6^{lm} \sin \theta W_{lm} & K_6^{lm} \sin \theta X_{lm} \end{pmatrix}. \quad (34)$$

Herein and throughout the whole paper, we use the shorthand notation

$$\Lambda := l(l+1). \quad (35)$$

We should note that the somewhat peculiar looking expansions for the coefficient  $K_5^{lm}$  can actually be written as

$$W_{lm} - \Lambda Y_{lm} = 2\partial_{\theta\theta}^2 Y_{lm}, \quad (36)$$

$$-\sin^2 \theta (W_{lm} + \Lambda Y_{lm}) = 2(\cos \theta \sin \theta \partial_\theta + \partial_{\phi\phi}^2) Y_{lm}, \quad (37)$$

which are essentially the diagonal terms of the Regge–Wheeler tensor harmonic  $\Psi_{\alpha\beta}^{lm}$  (c.f. Eq. (20) of Ruoff 2001a). However, we prefer to write them in terms of  $W_{lm}$  and  $Y_{lm}$  because it is only for these quantities that simple orthogonality relations apply. Furthermore, we have to mention that in the definition of the polar components of the extrinsic curvature, we differ from the notation of Ruoff (2001a), where the meaning of  $K_4$  and  $K_5$  is reversed (c.f. Eq. (24)). Also, the expansion for the axial perturbations is not exactly the same as in Ruoff & Kokkotas (2001a,b).

In their original paper, Battistoni et al. (1971) did not use the ADM formalism to fix the gauge, instead they defined their gauge by directly setting  $h_{tt}$ ,  $h_{\theta\theta}$ ,  $h_{\theta\phi}$  and  $h_{\phi\phi}$  to zero. Since the relation between  $h_{tt}$  and the lapse  $\alpha$  is given by

$$h_{tt} = 2A\alpha + 2B^i \beta_i = 2e^\nu \alpha - 2\omega h_{t\phi}, \quad (38)$$

it follows that in the rotating case,  $h_{tt} \neq 0$  although the lapse  $\alpha$  vanishes. In the non-rotating case  $\beta_i = 0$  and both  $\alpha$  and  $h_{tt}$  vanish. If we insisted on keeping a vanishing  $h_{tt}$  also in the rotating case, we would obtain a non-vanishing lapse, giving us undesired second order spatial derivatives in the perturbation equations.

Finally the fluid perturbations are decomposed as

$$\delta u_i^{polar} = -e^\nu \sum_{l,m} \left( u_1^{lm}, u_2^{lm} \partial_\theta, u_2^{lm} \partial_\phi \right) Y_{lm} , \quad (39)$$

$$\delta u_i^{axial} = -e^\nu \sum_{l,m} \left( 0, -u_3^{lm} \frac{\partial_\phi}{\sin \theta}, u_3^{lm} \sin \theta \partial_\theta \right) Y_{lm} , \quad (40)$$

$$\delta \epsilon = \sum_{l,m} \rho^{lm} Y_{lm} , \quad (41)$$

$$\delta p = (p + \epsilon) \sum_{l,m} H^{lm} Y_{lm} , \quad (42)$$

$$\xi^r = \left[ \nu' \left( 1 - \frac{\Gamma_1}{\Gamma} \right) \right]^{-1} \sum_{l,m} \xi^{lm} Y_{lm} . \quad (43)$$

From Eq. (21), we have the relation

$$\rho^{lm} = \frac{(p + \epsilon)^2}{\Gamma_1 p} (H^{lm} - \xi^{lm}) . \quad (44)$$

For the sake of notational simplicity, we will from now on omit the indices  $l$  and  $m$  for the perturbation variables. With the above expansion, the evolution equations for  $h_{ij}$  read:

$$\begin{aligned} (\partial_t + i m \omega) S_3 Y_{lm} &= (2S_2' + 2\lambda' S_2 - K_1) Y_{lm} \\ &\quad + 2\omega e^{-2\lambda} (V_3' - \lambda' V_3) \partial_\phi Y_{lm} + 2\omega e^{-2\lambda} (V_4' - \lambda' V_4) \sin \theta \partial_\theta Y_{lm} , \end{aligned} \quad (45)$$

$$\begin{aligned} \partial_t \left( V_3 \partial_\theta - V_4 \frac{\partial_\phi}{\sin \theta} \right) Y_{lm} &= \left( V_1' - \frac{2}{r} V_1 + e^{2\lambda} (S_2 - K_2) \right) \partial_\theta Y_{lm} - \left( V_2' - \frac{2}{r} V_2 - e^{2\lambda} K_3 \right) \frac{\partial_\phi Y_{lm}}{\sin \theta} \\ &\quad - \omega \Lambda V_4 \sin \theta Y_{lm} , \end{aligned} \quad (46)$$

$$\partial_t (V_3 \partial_\phi + V_4 \sin \theta \partial_\theta) Y_{lm} = \left( V_1' - \frac{2}{r} V_1 + e^{2\lambda} (S_2 - K_2) \right) \partial_\phi Y_{lm} + \left( V_2' - \frac{2}{r} V_2 - e^{2\lambda} K_3 \right) \sin \theta \partial_\theta Y_{lm} , \quad (47)$$

$$0 = \left( 2S_2 - \frac{\Lambda}{r} V_1 - K_4 + \frac{\Lambda}{r} K_5 \right) Y_{lm} + 2\omega e^{-2\lambda} (V_3 \partial_\phi Y_{lm} + V_4 \sin \theta \partial_\theta Y_{lm}) , \quad (48)$$

$$0 = (V_1 - K_5) W_{lm} + (V_2 - K_6) \frac{X_{lm}}{\sin \theta} , \quad (49)$$

$$0 = (V_1 - K_5) X_{lm} - (V_2 - K_6) \sin \theta W_{lm} . \quad (50)$$

Still, in every equation a sum over all  $l$  and  $m$  is implied. From Eqs. (49) and (50) we immediately obtain our condition for the shift components

$$V_1 = K_5 , \quad (51)$$

$$V_2 = K_6 , \quad (52)$$

and from Eq. (48) it follows after multiplication with  $Y_{lm}^*$  and integration over the 2-sphere that

$$S_2 = \frac{1}{2} K_4 - \omega e^{-2\lambda} (i m V_3 + \mathcal{L}_1^{\pm 1} V_4) , \quad (53)$$

where we have defined the operator  $\mathcal{L}_1^{\pm 1}$ , which couples the equations of order  $l$  to the equations of order  $l + 1$  and  $l - 1$ , as

$$\mathcal{L}_1^{\pm 1} A_{lm} := \sum_{l'm'} A_{l'm'} \int_{S_2} Y_{lm}^* \sin \theta \partial_\theta Y_{l'm'} d\Omega = (l - 1) Q_{lm} A_{l-1m} - (l + 2) Q_{l+1m} A_{l+1m} , \quad (54)$$

with

$$Q_{lm} := \sqrt{\frac{(l - m)(l + m)}{(2l - 1)(2l + 1)}} . \quad (55)$$

Later, we will also need

$$\mathcal{L}_2^{\pm 1} A_{lm} := \sum_{l'm'} A_{l'm'} \int_{S_2} \partial_\theta Y_{lm}^* \sin \theta Y_{l'm'} d\Omega = -(l + 1) Q_{lm} A_{l-1m} + l Q_{l+1m} A_{l+1m} \quad (56)$$

and

$$\begin{aligned} \mathcal{L}_3^{\pm 1} A_{lm} &:= \sum_{l'm'} A_{l'm'} \left( l'(l' + 1) \int_{S_2} Y_{lm}^* \cos \theta Y_{l'm'} d\Omega + \int_{S_2} Y_{lm}^* \sin \theta \frac{\partial}{\partial \theta} Y_{l'm'} d\Omega \right) \\ &= (l - 1)(l + 1) Q_{lm} A_{l-1m} + l(l + 2) Q_{l+1m} A_{l+1m} . \end{aligned} \quad (57)$$

The operator  $\mathcal{L}_3^{\pm 1}$  can actually be expressed in terms of  $\mathcal{L}_1^{\pm 1}$  and  $\mathcal{L}_2^{\pm 1}$ :

$$\mathcal{L}_3^{\pm 1} = -\frac{1}{2} \left( \mathcal{L}_1^{\pm 1} (\Lambda - 2) + \mathcal{L}_2^{\pm 1} \Lambda \right). \quad (58)$$

By making use of these relations we can eliminate the spherical harmonics and obtain the following simple set of evolution equations for the metric perturbations:

$$(\partial_t + im\omega) S_3 = K'_4 - K_1 + \lambda' K_4 - 2\omega' e^{-2\lambda} (imV_3 + \mathcal{L}_1^{\pm 1} V_4), \quad (59)$$

$$(\partial_t + im\omega) V_3 = K'_5 - e^{2\lambda} K_2 - \frac{2}{r} K_5 + \frac{1}{2} e^{2\lambda} K_4, \quad (60)$$

$$(\partial_t + im\omega) V_4 = K'_6 - e^{2\lambda} K_3 - \frac{2}{r} K_6. \quad (61)$$

In a similar way, we obtain the evolution equations for the six extrinsic curvature components, which are a little more lengthy:

$$\begin{aligned} (\partial_t + im\omega) K_1 = & e^{2\nu-2\lambda} \left[ \left( \nu' + \frac{2}{r} \right) S'_3 - 2\frac{\Lambda}{r^2} V'_3 + 2\lambda' \frac{\Lambda}{r^2} V_3 + 2 \left( \frac{\nu'}{r} - \frac{\lambda'}{r} - \frac{e^{2\lambda}-1}{r^2} + e^{2\lambda} \frac{\Lambda}{2r^2} \right) S_3 \right] \\ & + 8\pi e^{2\nu} (p + \epsilon) C_s^{-2} [(C_s^2 - 1) H + \xi] - 2e^{-2\lambda} \omega' \left[ im \left( K'_5 - \frac{2}{r} K_5 \right) + \mathcal{L}_1^{\pm 1} \left( K'_6 - \frac{2}{r} K_6 \right) \right], \end{aligned} \quad (62)$$

$$\begin{aligned} (\partial_t + im\omega) K_2 = & e^{2\nu-2\lambda} \left( \left( \nu' + \frac{1}{r} \right) S_3 - \frac{2}{r^2} V_3 \right) \\ & + \frac{imr^2}{2\Lambda} e^{-2\lambda} \left[ \omega' \left( K'_4 - K_1 + \lambda' K_4 - 4\frac{\Lambda-1}{r^2} K_5 \right) - 16\pi\varpi(p + \epsilon) (e^{2\lambda} K_4 + 2e^{2\nu} u_1) \right] \\ & - \frac{\omega' e^{-2\lambda}}{\Lambda} \mathcal{L}_1^{\pm 1} ((\Lambda - 2) K_6), \end{aligned} \quad (63)$$

$$\begin{aligned} (\partial_t + im\omega) K_3 = & e^{2\nu-2\lambda} \frac{\Lambda-2}{r^2} V_4 + e^{-2\lambda} \frac{\omega'}{\Lambda} (2imK_6 + (\Lambda - 2) \mathcal{L}_2^{\pm 1} K_5) \\ & - \frac{r^2}{2\Lambda} e^{-2\lambda} \mathcal{L}_2^{\pm 1} \left[ \omega' (K'_4 - K_1 + \lambda' K_4) - 16\pi\varpi(p + \epsilon) (e^{2\lambda} K_4 + 2e^{2\nu} u_1) \right], \end{aligned} \quad (64)$$

$$\begin{aligned} (\partial_t + im\omega) K_4 = & e^{2\nu-2\lambda} \left[ S'_3 + 2 \left( \nu' - \lambda' + \frac{1}{r} \right) S_3 - \frac{2\Lambda}{r^2} V_3 \right] + 8\pi r e^{2\nu} (p + \epsilon) C_s^{-2} [(C_s^2 - 1) H + \xi] \\ & + r (\mathcal{L}_1^{\pm 1} - \mathcal{L}_2^{\pm 1}) (\omega' K_3 + 16\pi e^{2\nu} \varpi(p + \epsilon) u_3), \end{aligned} \quad (65)$$

$$\begin{aligned} (\partial_t + im\omega) K_5 = & e^{2\nu-2\lambda} \left( V'_3 + (\nu' - \lambda') V_3 - \frac{1}{2} e^{2\lambda} S_3 \right) \\ & + \frac{r^2}{\Lambda} \left\{ im \left[ \omega' \left( \frac{1}{2} K_4 - K_2 \right) - 16\pi e^{2\nu} \varpi(p + \epsilon) u_2 \right] - \mathcal{L}_2^{\pm 1} (\omega' K_3 + 16\pi e^{2\nu} \varpi(p + \epsilon) u_3) \right\}, \end{aligned} \quad (66)$$

$$\begin{aligned} (\partial_t + im\omega) K_6 = & e^{2\nu-2\lambda} (V'_4 + (\nu' - \lambda') V_4) \\ & - \frac{r^2}{\Lambda} \left\{ im (\omega' K_3 + 16\pi e^{2\nu} \varpi(p + \epsilon) u_3) + \mathcal{L}_2^{\pm 1} \left[ \omega' \left( \frac{1}{2} K_4 - K_2 \right) - 16\pi e^{2\nu} \varpi(p + \epsilon) u_2 \right] \right\}. \end{aligned} \quad (67)$$

It is worthwhile to point out the symmetry between the polar and axial equations. Each pair  $V_3$  and  $V_4$ ,  $K_2$  and  $K_3$ , and  $K_5$  and  $K_6$  represent the polar and axial counterparts of a metric or extrinsic curvature perturbation. Thus, each associated pair of equations (60) and (61), (63) and (64), and (66) and (67) has basically the same structure, with only the polar equations containing additional terms as there are more polar variables than axial ones.

The last missing set of evolution equations is the one for the fluid quantities, coming from  $\delta(T^{\mu\nu}_{;\mu}) = 0$  and from Eq. (24):

$$\begin{aligned} (\partial_t + im\Omega) H = & C_s^2 \left\{ e^{2\nu-2\lambda} \left[ u'_1 + \left( 2\nu' - \lambda' + \frac{2}{r} \right) u_1 - e^{2\lambda} \frac{\Lambda}{r^2} u_2 \right] + \frac{1}{2} K_1 - \frac{\Lambda}{r^2} K_5 + \frac{1}{r} K_4 \right. \\ & \left. + \varpi e^{-2\lambda} \left[ im \left( V'_3 + \left( \frac{2}{r} - \lambda' \right) V_3 + e^{2\lambda} \left( H - \frac{1}{2} S_3 \right) \right) + \mathcal{L}_1^{\pm 1} \left( V'_4 + \left( \frac{2}{r} - \lambda' \right) V_4 \right) \right] \right\} \\ & - \nu' \left[ e^{2\nu-2\lambda} u_1 + \frac{1}{2} K_4 + \varpi e^{-2\lambda} (imV_3 + \mathcal{L}_1^{\pm 1} V_4) \right], \end{aligned} \quad (68)$$

$$\begin{aligned} (\partial_t + im\Omega) u_1 = & H' + \frac{p'}{\Gamma_{1p}} \left[ \left( \frac{\Gamma_1}{\Gamma} - 1 \right) H + \xi \right] - im \left[ e^{-2\nu} \varpi \left( K'_5 - \frac{2}{r} K_5 \right) + \left( \omega' + 2\varpi \left( \nu' - \frac{1}{r} \right) \right) u_2 \right] \\ & - \mathcal{L}_1^{\pm 1} \left[ e^{-2\nu} \varpi \left( K'_6 - \frac{2}{r} K_6 \right) + \left( \omega' + 2\varpi \left( \nu' - \frac{1}{r} \right) \right) u_3 \right], \end{aligned} \quad (69)$$

$$(\partial_t + im\Omega) u_2 = H + \frac{\varpi}{\Lambda} (im(2u_2 - e^{-2\nu} (\Lambda - 2) K_5) + 2\mathcal{L}_3^{\pm 1} u_3 - e^{-2\nu} \mathcal{L}_1^{\pm 1} ((\Lambda - 2) K_6)) - \frac{imr^2}{\Lambda} A, \quad (70)$$

$$(\partial_t + im\Omega) u_3 = 2\frac{\varpi}{\Lambda} \left[ im(u_3 + e^{-2\nu} K_6) - \mathcal{L}_3^{\pm 1}(u_2 + e^{-2\nu} K_5) \right] + \frac{r^2}{\Lambda} \mathcal{L}_2^{\pm 1} A, \quad (71)$$

$$(\partial_t + im\Omega) \xi = \nu' \left( \frac{\Gamma_1}{\Gamma} - 1 \right) \left[ e^{2\nu-2\lambda} u_1 + \frac{1}{2} K_4 + \varpi e^{-2\lambda} (imV_3 + \mathcal{L}_1^{\pm 1} V_4) \right], \quad (72)$$

where

$$A = \varpi C_s^2 \left\{ e^{-2\lambda} \left[ u_1' + \left( 2\nu' - \lambda' + \frac{2}{r} \right) u_1 - e^{2\lambda} \frac{\Lambda}{r^2} u_2 \right] + e^{-2\nu} \left[ \frac{1}{2} K_1 - \frac{\Lambda}{r^2} K_5 + \frac{1}{r} K_4 \right] \right\} \\ + \left[ \varpi \left( \nu' - \frac{2}{r} \right) + \omega' \right] \left( e^{-2\lambda} u_1 + \frac{1}{2} e^{-2\nu} K_4 \right). \quad (73)$$

In Eq. (68), we have defined the sound speed  $C_s$  as

$$C_s^2 = \frac{\Gamma_1}{\Gamma} \frac{dp}{d\epsilon}. \quad (74)$$

The evolution equations comprise fourteen equations in total: four axial and ten polar ones. In the non-rotating case, they are equivalent to four wave equations, one for the axial and two for the polar metric perturbations plus one wave equation for the fluid variable  $H$ . The fluid equation for the axial velocity perturbation  $u_3$  vanishes in the non-rotating case, whereas equation (72) for the displacement variable  $\xi$  does so in the barotropic case.

Finally we have the four constraint equations. The Hamiltonian constraint reads

$$8\pi r^2 e^{2\lambda} \rho = rS_3' - \Lambda V_3' + \left( 1 - 2r\lambda' + \frac{1}{2} e^{2\lambda} \Lambda \right) S_3 + \Lambda \left( \lambda' - \frac{1}{r} \right) V_3 \\ + r^2 e^{2\lambda} \left[ im \left( \frac{1}{2} \omega' e^{-2\nu} K_2 + 16\pi\varpi(p+\epsilon)u_2 \right) + \mathcal{L}_1^{\pm 1} \left( \frac{1}{2} \omega' e^{-2\nu} K_3 + 16\pi\varpi(p+\epsilon)u_3 \right) \right], \quad (75)$$

and the three momentum constraints are

$$8\pi r e^{2\nu} (p+\epsilon) u_1 = K_4' - \frac{\Lambda}{r} K_5' - K_1 + e^{2\lambda} \frac{\Lambda}{2r} K_2 + \frac{\Lambda}{r^2} (1 + r\nu') K_5 - \nu' K_4 + \frac{im}{4} r \omega' S_3 \\ - (8\pi r (p+\epsilon) \varpi + 2e^{-2\lambda} \omega') (imV_3 + \mathcal{L}_1^{\pm 1} V_4), \quad (76)$$

$$16\pi r e^{2\nu} (p+\epsilon) u_2 = -rK_2' + rK_1 + (r\nu' - r\lambda' - 2) K_2 - \frac{2}{r} K_5 + K_4 + e^{-2\lambda} \frac{r\omega'}{\Lambda} (2imV_3 - (\Lambda - 2) \mathcal{L}_2^{\pm 1} V_4) \\ + \frac{imr^3}{\Lambda} \left[ \frac{1}{2} e^{-2\lambda} \omega' S_3' - 16\pi\varpi(p+\epsilon) (S_3 + C_s^{-2} ((C_s^2 + 1) H - \xi)) \right], \quad (77)$$

$$16\pi r e^{2\nu} (p+\epsilon) u_3 = -rK_3' + (r\nu' - r\lambda' - 2) K_3 + \frac{\Lambda - 2}{r} K_6 + e^{-2\lambda} \frac{r\omega'}{\Lambda} (2imV_4 + (\Lambda - 2) \mathcal{L}_2^{\pm 1} V_3) \\ - \frac{r^3}{\Lambda} \mathcal{L}_2^{\pm 1} \left[ \frac{1}{2} e^{-2\lambda} \omega' S_3' - 16\pi\varpi(p+\epsilon) (S_3 + C_s^{-2} ((C_s^2 + 1) H - \xi)) \right]. \quad (78)$$

The axial equations (61), (64), (67), (71) and (78) without the coupling terms to the polar perturbations are equivalent to Eqs. (7)–(10) and (12) of Ruoff & Kokkotas (2001b). Note, however, that therein a slightly different definition of the perturbation variables has been chosen.

### 3 THE NON-ROTATING LIMIT

Although the non-rotating limit is well described by the wave equations given by Allen et al. (1998), it is instructive to consider it in the BCL gauge. This is obtained by setting  $\Omega$  and  $\omega$  to zero in all the evolution equations (59)–(72) and the constraints (75)–(78). As is well known, in this case the polar and axial parts of the equations completely decouple. For barotropic perturbations ( $\Gamma_1 = \Gamma$ ), the polar evolution equations can then be easily transformed into three wave equations for the rescaled metric variables  $S = e^{\nu-\lambda} S_3$  and  $V = e^{\nu-\lambda} V_3/r$  and the rescaled fluid variable  $\tilde{H} = e^{-\nu-\lambda} H/r$ :

$$\frac{\partial^2 S}{\partial t^2} = \frac{\partial^2 S}{\partial r_*^2} + e^{2\nu-2\lambda} \left[ \left( \nu' (\nu' - \lambda') + 3\frac{\nu'}{r} + \frac{\lambda'}{r} - \frac{3}{r^2} - e^{2\lambda} \frac{\Lambda - 1}{r^2} - \lambda'' \right) S + \frac{4\Lambda}{r^2} (1 - r\nu') V \right] \\ + 8\pi e^{2\nu-2\lambda} \left[ (C_s^2 - 1) \left( \tilde{\rho}' + \left( \nu' - \frac{1}{r} \right) \tilde{\rho} \right) + (C_s^2)' \tilde{\rho} \right], \quad (79)$$

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial r_*^2} + e^{2\nu-2\lambda} \left[ \left( \frac{\nu'}{r} - \frac{\lambda'}{r} + 2\frac{e^{2\lambda}-1}{r^2} - e^{2\lambda} \frac{\Lambda}{r^2} \right) V - e^{2\lambda} \left( \frac{\nu'}{r} + \frac{\lambda'}{r} - \frac{1}{r^2} \right) S \right] \\ + 4\pi e^{2\nu} (C_s^2 - 1) \tilde{\rho}, \quad (80)$$

$$\begin{aligned}
 \frac{\partial^2 \tilde{H}}{\partial t^2} = & e^{2\nu-2\lambda} \left\{ C_s^2 \frac{\partial^2 \tilde{H}}{\partial r^2} - (C_s^2 \lambda' + \nu') \frac{\partial \tilde{H}}{\partial r} \right. \\
 & + \left[ C_s^2 \left( \lambda' \left( \frac{3}{r} + \lambda' \right) + \frac{e^{2\lambda} - 1}{r^2} - \lambda'' - e^{2\lambda} \frac{\Lambda}{r^2} \right) + \frac{\lambda'}{r} + 2 \frac{\nu'}{r} + \frac{\nu'}{C_s^2} (\nu' + \lambda') \right] \tilde{H} \Big\} \\
 & + e^{2\nu} \left\{ \frac{r\nu'}{2} (C_s^2 - 1) \frac{\partial S}{\partial r} + \left[ C_s^2 \left( \frac{\nu'}{2} (r\lambda' - r\nu' + 6) + \lambda' - \frac{e^{2\lambda} - 1}{r} \right) + \frac{\nu'}{2} (r\lambda' - r\nu' - 2) \right] S \right. \\
 & \left. \left. - \nu' \Lambda (C_s^2 - 1) V \right\} . \tag{81}
 \end{aligned}$$

In Eqs. (79) and (80),  $r_*$  is the well-known tortoise coordinate, which is related to  $r$  through

$$\frac{d}{dr_*} = e^{\nu-\lambda} \frac{d}{dr} . \tag{82}$$

Furthermore, one can express the energy density  $\tilde{\rho}$  in terms of  $\tilde{H}$ , which in the barotropic case reduces to

$$\tilde{\rho} = \frac{p + \epsilon}{C_s^2} \tilde{H} . \tag{83}$$

Although the equations in the first order form are quite simple, the above set of wave equations is more complicated than the equivalent set in the Regge–Wheeler gauge (Eqs. (14), (15) and (16) of Allen et al. 1998). This is particular so for the way in which the fluid variable  $\tilde{\rho}$  (or equivalently  $\tilde{H}$ ) couples to the metric variable  $S$ , where the derivative of  $\tilde{\rho}$  enters. If the stellar model is based on a polytropic equation of state  $p = \kappa \epsilon^\Gamma$ , then the behaviour of  $\tilde{\rho}$  at the stellar surface strongly depends on the polytropic index  $\Gamma$ . As discussed in Ruoff (2001a),  $\tilde{\rho}$  actually diverges for  $\Gamma > 2$ . In this case the metric quantity  $S$  would not even be  $\mathcal{C}^0$ , which can be troublesome for the numerical convergence. Although this could be a drawback of the BCL gauge, there is a clear advantage if one is interested in computing the gauge invariant Zerilli function  $Z$  in the exterior. Following Moncrief (1974), the definition of the Zerilli function is

$$Z = \frac{r^2 (\Lambda k_1 + 4e^{-4\lambda} k_2)}{r (\Lambda - 2) + 2M} \tag{84}$$

with

$$k_1 = -2e^{-\lambda-\nu} V \tag{85}$$

and

$$k_2 = \frac{1}{2} e^{3\lambda-\nu} S . \tag{86}$$

In terms of  $S$  and  $V$  this gives us

$$Z = \frac{2r^2 e^{-\lambda-\nu}}{r (\Lambda - 2) + 2M} (S - \Lambda V) , \tag{87}$$

which is a simple algebraic relation in contrast to the relation in the Regge–Wheeler gauge, which includes a spatial derivative of one of the metric perturbations (see Eq. (20) of Allen et al. 1998, or Eq. (60) of Ruoff 2001a). In the Regge–Wheeler gauge, the two metric variables ( $S$  and  $F$  in the notation of Allen et al. 1998, and  $S$  and  $T$  in the notation of Ruoff 2001a) have different asymptotic behaviour at infinity, in particular one ( $F$  or  $T$ ) is linearly growing with  $r$ . It is only through the delicate cancellation of the growing terms that the Zerilli function remains finite at infinity. However, this cancellation can only happen if both metric variables exactly satisfy the Hamiltonian constraint. Any (numerical) violation leads to an incomplete cancellation, and the Zerilli function starts to grow at large radii. This makes it very difficult in the numerical time evolution to extract the correct amount of gravitational radiation emitted from the neutron star. With the above relation (87), we do not expect such difficulties to occur.

## 4 CONCLUSIONS

We have presented the derivation of the perturbation equations for slowly rotating relativistic stars using the BCL gauge, which has been first used by Battiston, Cazzola & Lucaroni in 1971. This gauge is defined by setting  $\alpha$ ,  $h_{\theta\theta}$ ,  $h_{\theta\phi}$  and  $h_{\phi\phi}$  to zero. In the non-rotating case, the vanishing shift condition leads to a complete vanishing of  $h_{tt}$ , however, in the rotating case  $h_{tt}$  becomes non-zero (see also Appendix). The advantage of the BCL gauge over the Regge–Wheeler gauge is that in the ADM formalism, the evolution equations a priori do not contain any second order spatial derivatives. Instead, one is immediately lead to a hyperbolic set of first order evolution equations, which can be directly used for the numerical time evolution without many further manipulations. Although it is in principle also possible to derive a hyperbolic set in the Regge–Wheeler gauge,

the procedure is rather tedious and requires the introduction of carefully chosen new variables in order to replace the second order or mixed derivatives.

The perturbation equations for slowly rotating relativistic stars form a set of fourteen evolution equations plus four constraints. In the non-rotating barotropic case, it is possible to cast the equations into a system of four wave equations, three for the polar and axial metric perturbations and one for the polar fluid perturbation, as it is the case in the Regge–Wheeler gauge. Although these wave equations are not simpler than the corresponding ones in the Regge–Wheeler gauge, the first order system actually is. Maybe the main advantage is the simple algebraic relation of the metric variables to the Zerilli function. It was demonstrated by Ruoff (2001a) that the accurate numerical evaluation of the Zerilli function in the Regge–Wheeler gauge is somewhat difficult and requires high resolution because a small numerical violation of the Hamiltonian constraint can lead to very large errors in the Zerilli function. This should not be the case in the BCL gauge as the relation (87) does not involve any derivatives.

A further advantage of these evolution equations is that the inclusion of the source terms describing a particle orbiting the star can be very easily accomplished. This is not the case for the Regge–Wheeler gauge, as even for the non-rotating case, one is forced to include second order derivatives of the source terms (Ruoff 2001b). Since the source terms contain  $\delta$ -functions, one has to deal with second order derivatives thereof. In the axial case, no derivatives appear, and the perturbation equations with the source terms, which are given in Ruoff (2001b), are very simple. We expect the same to be the case for the polar equations in the BCL gauge. Here, it should be possible to plug the source terms into the equations for the extrinsic curvature without getting any derivatives.

In subsequent papers, we will present results from the numerical evolution of the perturbation equations of slowly rotating relativistic stars in the BCL gauge with the particular focus on oscillations modes which are unstable with respect to gravitational radiation. As a step further, we will include the contribution of a test particle acting as a source of excitation for the stellar oscillations.

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# APPENDIX: THE PERTURBATION EQUATIONS FOLLOWING FROM EINSTEIN'S EQUATIONS

Kojima (1992) derived the perturbation equations in the Regge–Wheeler gauge directly from the linearized Einstein equations without resorting to the ADM formalism. In this section we repeat this calculation using the BCL gauge. In order to facilitate the comparison with Kojima's equations, who uses the more familiar notation of Regge & Wheeler (1957), we switch to a similar notation. In the Regge–Wheeler gauge, the quantities  $h_0$  and  $h_1$  denote the axial perturbations of  $h_{t\{\phi,\theta\}}$  and  $h_{r\{\phi,\theta\}}$ , respectively, whereas the corresponding polar perturbations are set to zero. Since in the BCL gauge, the latter do not vanish, we denote them by  $h_{0,p}$  and  $h_{1,p}$ , respectively, and to avoid confusion we denote the axial ones by  $h_{0,a}$  and  $h_{1,a}$ . The remaining non-zero polar perturbations are then  $H_1$  and  $H_2$ . Thus, the expansion of the metric in BCL gauge reads in this notation:

$$h_{\mu\nu} = \sum_{lm} \begin{pmatrix} -2\omega (h_{0,p}^{lm} \partial_\phi + h_{0,a}^{lm} \sin\theta \partial_\theta) & H_1^{lm} & h_{0,p}^{lm} \partial_\theta - h_{0,a}^{lm} / \sin\theta \partial_\phi & h_{0,p}^{lm} \partial_\phi + h_{0,a}^{lm} \sin\theta \partial_\theta \\ \star & e^{2\lambda} H_2^{lm} & h_{1,p}^{lm} \partial_\theta - h_{1,a}^{lm} / \sin\theta \partial_\phi & h_{1,p}^{lm} \partial_\phi + h_{1,a}^{lm} \sin\theta \partial_\theta \\ \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \end{pmatrix} Y_{lm} . \quad (88)$$

Note that here  $h_{tt}$  is not zero, which is a consequence of the relation between the perturbation of the lapse  $\alpha$  and  $h_{tt}$  given by Eq. (38). The relation between the above variables and the ones used in the previous sections is the following (we again omit the indices  $l$  and  $m$ ):

$$H_1 = e^{2\lambda} K_4 - \omega (imV_3 + \mathcal{L}_1^{\pm 1} V_4) , \quad (89)$$

$$H_2 = S_3 , \quad (90)$$

$$h_{0,p} = K_5 , \quad (91)$$

$$h_{0,a} = K_6 , \quad (92)$$

$$h_{1,p} = V_3 , \quad (93)$$

$$h_{1,a} = V_4 , \quad (94)$$

$$R = -u_1 , \quad (95)$$

$$V = -u_2 , \quad (96)$$

$$U = -u_3 . \quad (97)$$

The extrinsic curvature components can be expressed as

$$K_1 = 2e^{-2\lambda} \{ H_1' - \lambda' H_1 + \omega [im(h_{1,p}' - \lambda' h_{1,p}) + \mathcal{L}_1^{\pm 1} (h_{1,a}' - \lambda' h_{1,a})] \} - \dot{H}_2 - im\omega H_2 , \quad (98)$$

$$K_2 = e^{-2\lambda} \left[ h_{0,p}' - \frac{2}{r} h_{0,p} + H_1 - \dot{h}_{1,p} + \omega \mathcal{L}_1^{\pm 1} h_{1,a} \right] , \quad (99)$$

$$K_3 = e^{-2\lambda} \left[ h_{0,a}' - \frac{2}{r} h_{0,a} + \dot{h}_{1,a} - im\omega h_{1,a} \right] . \quad (100)$$

A very often occurring combination of variables in the perturbation equations is  $h_0' - \dot{h}_1$  for both the axial and polar cases, which we abbreviate with the following functions

$$Z_a = h_{0,a}' - \dot{h}_{1,a} , \quad (101)$$

$$Z_p = h_{0,p}' - \dot{h}_{1,p} . \quad (102)$$

The equations coming from the  $(tt)$ ,  $(tr)$ ,  $(rr)$  and the addition of the  $(\theta\theta)$  and  $(\phi\phi)$  components can be written as

$$A_{lm}^{(I)} + imC_{lm}^{(I)} + \mathcal{L}_2^{\pm 1} B_{lm}^{(I)} + \mathcal{L}_4^{\pm 1} \tilde{A}_{lm}^{(I)} = 0 , \quad (103)$$

with

$$\mathcal{L}_4^{\pm 1} A_{lm} := -\frac{1}{2} (\mathcal{L}_1^{\pm 1} + \mathcal{L}_2^{\pm 1}) A_{lm} = Q_{lm} A_{l-1m} + Q_{l+1m} A_{l+1m} \quad (104)$$

and

$$A^{(tt)} = \frac{2e^{2\nu}}{r^2} \left[ rH_2' - \Lambda h_{1,p}' - 16\pi r^2 e^{2\lambda} C_s^{-2} (H - \xi) + \Lambda \left( \lambda' - \frac{1}{r} \right) h_{1,p} + \left( 1 - 2r\lambda' + \frac{\Lambda e^{2\lambda}}{2} \right) H_2 \right] , \quad (105)$$

$$\tilde{A}^{(tt)} = 0 , \quad (106)$$

$$B^{(tt)} = 2\omega Z_a' + \left( \omega' - 2\omega \left( \lambda' + \nu' - \frac{2}{r} \right) \right) Z_a - \frac{4\omega}{r} h_{0,a}' - 32\pi\Omega (p + \epsilon) e^{2\nu+2\lambda} U \\ + \frac{2}{r} \left[ -\omega' + \omega \left( 2\nu' + 2\lambda' - \frac{2}{r} - e^{2\lambda} \frac{\Lambda - 2}{r} \right) \right] h_{0,a} , \quad (107)$$

$$C^{(tt)} = 2\omega (Z_p' - H_1' + e^{2\lambda} \dot{H}_2) + \left( \omega' - 2\omega (\lambda' + \nu') \right) h_{0,p}' - \frac{2}{r} \left[ \omega' - 2\omega \left( \nu' + \lambda' - \frac{e^{2\lambda} - 1}{r} \right) \right] h_{0,p}$$

$$- \left[ \omega' - 2\omega \left( \nu' + \lambda' - \frac{2}{r} \right) \right] \dot{h}_{1,p} + (\omega' + 2\omega(\lambda' - \nu')) H_1 - 32\pi\Omega e^{2\nu+2\lambda} (p + \epsilon) V , \quad (108)$$

$$A^{(tr)} = \frac{2}{r} \dot{H}_2 + \frac{\Lambda}{r^2} (Z_p + H_1 - 2h'_{0,p} + 2\nu' h_{0,p}) + 16\pi (p + \epsilon) (e^{2\nu} R - H_1) , \quad (109)$$

$$\tilde{A}^{(tr)} = \frac{2\Lambda\omega}{r^2} h_{1,a} , \quad (110)$$

$$B^{(tr)} = \left[ \frac{\Lambda\omega}{r^2} - 16\pi\Omega (p + \epsilon) \right] h_{1,a} , \quad (111)$$

$$C^{(tr)} = \left( \frac{2\omega}{r} + \frac{\omega'}{2} \right) H_2 - 16\pi\Omega (p + \epsilon) h_{1,p} , \quad (112)$$

$$A^{(rr)} = \dot{H}_1 + e^{2\nu} \frac{\Lambda}{2r} \left( \lambda' + \frac{1}{r} \right) h_{1,p} - 4\pi r e^{2\nu+2\lambda} (p + \epsilon) H - \frac{e^{2\nu}}{2r} \left[ (2r\nu' + 1) - \frac{\Lambda}{2} e^{2\lambda} \right] H_2 - e^{2\nu} \frac{\Lambda}{2r} h'_{1,p} , \quad (113)$$

$$\tilde{A}^{(rr)} = 0 , \quad (114)$$

$$B^{(rr)} = \omega h'_{0,a} + \frac{\omega'}{2} h_{0,a} - \left( \omega + \frac{r\omega'}{4} \right) Z_a , \quad (115)$$

$$C^{(rr)} = \omega h'_{0,p} - \left( \omega + \frac{r\omega'}{4} \right) Z_p + \left( \frac{\omega'}{2} - e^{2\lambda} \frac{\Lambda\omega}{r} \right) h_{0,p} + \left( \omega + \frac{r\omega'}{4} \right) H_1 , \quad (116)$$

$$\begin{aligned} A^{(\theta\theta+\phi\phi)} &= -\ddot{H}_2 + 2e^{-2\lambda} \left[ \dot{H}'_1 + \left( \frac{1}{r} - \lambda' \right) \dot{H}_1 \right] - e^{2\nu-2\lambda} \left( \nu' + \frac{1}{r} \right) H'_2 - \frac{\Lambda}{r^2} (\dot{h}_{0,p} - e^{2\nu-2\lambda} h'_{1,p}) \\ &\quad - 16\pi e^{2\nu} (p + \epsilon) H - e^{2\nu} \left( \frac{\Lambda}{2r^2} + 16\pi p \right) H_2 + \frac{\Lambda}{r^2} e^{2\nu-2\lambda} (\nu' - \lambda') h_{1,p} , \end{aligned} \quad (117)$$

$$\tilde{A}^{(\theta\theta+\phi\phi)} = 0 , \quad (118)$$

$$B^{(\theta\theta+\phi\phi)} = 2\omega e^{-2\lambda} \left( h''_{0,a} - Z'_a + \left( \frac{1}{r} - \lambda' \right) (h_{0,a} - Z_a) \right) + 2\omega' e^{-2\lambda} \left( h'_{0,a} - \frac{2}{r} h_{0,a} \right) - 16\pi e^{2\nu} \varpi (p + \epsilon) U , \quad (119)$$

$$\begin{aligned} C^{(\theta\theta+\phi\phi)} &= 2\omega e^{-2\lambda} \left[ h''_{0,p} - Z'_p + H'_1 - e^{2\lambda} \left( \dot{H}_2 + \frac{\Lambda}{r^2} h_{0,p} \right) + \left( \frac{1}{r} - \lambda' \right) (\dot{h}_{1,p} + H_1) \right] \\ &\quad + e^{-2\lambda} \omega' \left( H_1 + 2h'_{0,p} - \frac{4}{r} h_{0,p} \right) - 16\pi \varpi e^{2\nu} (p + \epsilon) V . \end{aligned} \quad (120)$$

The  $(t\theta)$  and  $(r\theta)$  components are

$$\Lambda a_{lm}^{(I)} + i m d_{lm}^{(I)} + \mathcal{L}_3^{\pm 1} \tilde{a}_{lm}^{(I)} + \mathcal{L}_2^{\pm 1} \eta_{lm}^{(I)} = 0 , \quad (121)$$

with

$$\begin{aligned} a^{(t\theta)} &= -\dot{H}_2 + e^{-2\lambda} \left[ H'_1 - Z'_p + \frac{2}{r} h'_{0,p} + \left( \lambda' + \nu' - \frac{2}{r} \right) Z_p + (\nu' - \lambda') H_1 - \frac{2}{r^2} (r\lambda' - r\nu' + e^{2\lambda} - 1) h_{0,p} \right] \\ &\quad + 16\pi e^{2\nu} (p + \epsilon) V , \end{aligned} \quad (122)$$

$$d^{(t\theta)} = e^{-2\lambda} \left[ 2\Lambda\omega \left( h'_{1,p} + \left( \frac{1}{r} - \nu' \right) h_{1,p} \right) + \omega' \left( \frac{r^2}{2} H'_2 + 2h_{1,p} \right) \right] - 16\pi r^2 \varpi (p + \epsilon) (H_2 + (1 + C_s^{-2}) H + C_s^{-2} \xi) , \quad (123)$$

$$\tilde{a}^{(t\theta)} = 2\omega' e^{-2\lambda} h_{1,a} , \quad (124)$$

$$\eta^{(t\theta)} = -\Lambda\omega e^{-2\lambda} [h'_{1,a} + (\nu' - \lambda') h_{1,a}] , \quad (125)$$

$$\begin{aligned} a^{(r\theta)} &= -\dot{Z}_p + \frac{1}{r} \left( 2e^{2\nu-2\lambda} - \frac{\Lambda}{2} \right) h'_{1,p} + \left[ 8\pi e^{2\nu} (p + \epsilon) + \frac{\Lambda}{2r^2} (1 + r\lambda') - \frac{2}{r^2} e^{2\nu} \right] h_{1,p} \\ &\quad + \left[ \nu' (e^{2\nu} - 1) + \frac{1}{2r} \left( \frac{\Lambda}{2} e^{2\lambda} - 1 \right) \right] H_2 - 4\pi r e^{2\lambda} (p + \epsilon) H , \end{aligned} \quad (126)$$

$$d^{(r\theta)} = 16\pi r^2 \varpi (p + \epsilon) (H_1 + e^{2\nu} R) - \omega\Lambda (H_1 + Z_p - h_{0,p}) + \omega' \left( \frac{r^2}{2} \dot{H}_2 - 2(\Lambda + 2) h_{0,p} \right) , \quad (127)$$

$$\tilde{a}^{(r\theta)} = 2\omega' h_{0,a} , \quad (128)$$

$$\eta^{(r\theta)} = \Lambda [\omega (h'_{0,a} - Z_a) + \omega' h_{0,a}] . \quad (129)$$

From the  $(t\phi)$  and  $(r\phi)$  components we get

$$\Lambda b_{lm}^{(I)} + i m c_{lm}^{(I)} + \mathcal{L}_3^{\pm 1} \tilde{b}_{lm}^{(I)} + \mathcal{L}_2^{\pm 1} \zeta_{lm}^{(I)} = 0 , \quad (130)$$

with

$$b^{(t\phi)} = -Z'_a + \left( \nu' + \lambda' - \frac{2}{r} \right) Z_a + \frac{2}{r} h'_{0,a} + 16\pi e^{2\nu+2\lambda} (p + \epsilon) U - \left[ \frac{2}{r} \left( \nu' + \lambda' - \frac{1}{r} \right) - e^{2\lambda} \frac{\Lambda - 2}{r^2} \right] h_{0,a} , \quad (131)$$

$$c^{(t\phi)} = -3\Lambda\omega h'_{1,a} + \left[ \Lambda\omega \left( 3\lambda' - \nu' - \frac{2}{r} \right) - (\Lambda - 2)\omega' \right] h_{1,a} , \quad (132)$$

$$\tilde{b}^{(t\phi)} = -2e^{-2\lambda}\omega'h_{1,p}, \quad (133)$$

$$\zeta^{(t\phi)} = 2\omega\Lambda\left(e^{2\lambda}H_2 + (\nu' - \lambda')h_{1,p}\right) + \omega'\left(\Lambda h_{1,p} - \frac{r^2}{2}H_2'\right) + 16\pi r^2 e^{2\lambda}\varpi(p + \epsilon)\left[H_2 + (1 + C_s^{-2})H + C_s^{-2}\xi\right], \quad (134)$$

$$b^{(r\phi)} = \dot{Z}_a - \frac{2}{r}e^{2\nu-2\lambda}h'_{1,a} - e^{2\nu}\left[\frac{\Lambda-2}{r^2} + \frac{2}{r}e^{-2\lambda}(\nu' - \lambda')\right]h_{1,a}, \quad (135)$$

$$c^{(r\phi)} = \Lambda\omega h'_{0,a} + 2\left[(\Lambda + 1)\omega' - \frac{\Lambda\omega}{r}\right]h_{0,a}, \quad (136)$$

$$\tilde{b}^{(r\phi)} = 2\omega'h_{0,p}, \quad (137)$$

$$\zeta^{(r\phi)} = \omega'\left(\Lambda h_{0,p} - \frac{r^2}{2}\dot{H}_2\right) - 16\pi r^2\varpi(p + \epsilon)(e^{2\nu}R + H_1). \quad (138)$$

From the  $(\theta\phi)$  and the subtraction of  $(\theta\theta)$  and  $(\phi\phi)$  components we get

$$\Lambda s_{lm} - imf_{lm} + \mathcal{L}_2^{\pm 1}g_{lm} = 0, \quad (139)$$

$$\Lambda t_{lm} + img_{lm} + \mathcal{L}_2^{\pm 1}f_{lm} = 0, \quad (140)$$

with

$$f = \omega'r^2e^{-2\lambda}\left(Z_p - \frac{2}{r}h_{0,p}\right) - 16\pi r^2\varpi e^{2\nu}(p + \epsilon)V, \quad (141)$$

$$g = -\omega'r^2e^{-2\lambda}\left(Z_a - \frac{2}{r}h_{0,a}\right) + 16\pi r^2\varpi e^{2\nu}(p + \epsilon)U, \quad (142)$$

$$s = -\dot{h}_{0,p} + e^{2\nu-2\lambda}\left(h'_{1,p} + (\nu' - \lambda')h_{1,p}\right) - \frac{e^{2\nu}}{2}H_2 - im\omega h_{0,p}, \quad (143)$$

$$t = -\dot{h}_{0,a} + e^{2\nu-2\lambda}\left(h'_{1,a} + (\nu' - \lambda')h_{1,a}\right) - im\omega h_{0,a}. \quad (144)$$

These equations are fully equivalent to the ones derived within the ADM formalism. Although they still contain some second order derivatives, they can be easily brought by introduction of a few auxiliary variables into a first order hyperbolic form or even into characteristic form, which is very useful for the numerical evolution.